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Dissipative eigenvalue problems for a Sturm–Liouville operator with a singular potential*

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In this paper we consider the Sturm–Liouville operator $d^2/dx^2 - 1/x$ on the interval $[a, b]$, $a < 0 < b$, with Dirichlet boundary conditions at a and b , for which $x = 0$ is a singular point. In the two components $\mathcal{L}^2(a, 0)$ and $\mathcal{L}^2(0, b)$ of the space $\mathcal{L}^2(a, b) = \mathcal{L}^2(a, 0) \oplus \mathcal{L}^2(0, b)$ we define minimal symmetric operators and describe all the maximal dissipative and self-adjoint extensions of their orthogonal sum in $\mathcal{L}^2(a, b)$ by interface conditions at $x = 0$. We prove that the maximal dissipative extensions whose domain contains only continuous functions f are characterized by the interface condition $\lim_{x \rightarrow 0+} (f'(x) - f'(-x)) = \gamma f(0)$ with $\gamma \in \mathbb{C}^+ \cup \mathbb{R}$ or by the Dirichlet condition $f(0+) = f(0-) = 0$. We also show that the corresponding operators can be obtained by norm resolvent approximation from operators where the potential $1/x$ is replaced by a continuous function, and that their eigen and associated functions can be chosen to form a Bari basis in $\mathcal{L}^2(a, b)$.

1. Introduction

In this paper we consider the differential expression

$$l[f](x) := -f''(x) - \frac{f(x)}{x} \tag{1.1}$$

and the corresponding differential equation

$$-f''(x) - \frac{f(x)}{x} - \lambda f(x) = 0 \tag{1.2}$$

*Dedicated to Professor Boele Braaksma on the occasion of his 65th birthday, in friendship.

on the interval $[a, b]$, where $a < 0 < b$, with the boundary conditions

$$f(a) = f(b) = 0. \quad (1.3)$$

Since the potential is not summable at $x = 0$, it is not a classical Sturm–Liouville problem. We associate with this boundary eigenvalue problem two minimal operators in the spaces $\mathcal{L}^2([a, 0))$ and $\mathcal{L}^2((0, b])$. Since these operators are in the limit case at $x = 0$, they are not self-adjoint and their direct sum operator S in the space $\mathcal{L}^2([a, b])$ is symmetric with defect index $(2, 2)$. It is the aim of this paper to describe all self-adjoint and maximal dissipative extensions of S in $\mathcal{L}^2([a, b])$. Recall that an operator A in some Hilbert space \mathcal{H} is called *dissipative* if $\operatorname{Im}(Af, f) \geq 0$ for all $f \in \mathcal{H}$ and *maximal dissipative* if it does not have a proper dissipative extension. In particular, we also describe those extensions among them for which the domain consists only of continuous functions. This set turns out to be a one-parameter family of operators T_γ , $\gamma \in \mathbb{C}^+ \cup \{\infty\}$, which are defined by the interface condition

$$\lim_{x \rightarrow 0+} (f'(x) - f'(-x)) = \gamma f(0) \quad \text{if } \gamma \in \mathbb{C},$$

and by

$$f(0+) = f(0-) = 0 \quad \text{if } \gamma = \infty.$$

The problem (1.1) has been studied by several authors [4, 8, 12]. In [4] the potential $-x^{-1}$ is replaced by the regular potential $-(x - i\varepsilon)^{-1}$ and the resulting operator for $\varepsilon \rightarrow 0$ is considered. This operator is the extension T_γ with $\gamma = -i\pi$ (see Remark 5.2). In [8] the operator T_∞ is studied: it is the direct sum of two self-adjoint operators on $[a, 0)$ and $(0, b]$, respectively, with Dirichlet boundary conditions. Gunson treats the operators $T_{-i\pi}$ [12, theorem 2.6 and eqn (2.13)] and T_∞ [12, theorem 2.2 and eqn (2.1)] as well as T_0 , where the potential $-x^{-1}$ is considered in the distributional sense as the Cauchy principal value [12, theorem 2.4 and eqn (2.9)]. This self-adjoint operator is also studied in [1] from the viewpoint of quasi-derivatives. We mention that the operators T_γ considered here have discrete spectrum. The case where the interval $[a, b]$ is replaced by the real axis is also considered in [12]. In this case the corresponding operators $T_{i\theta}$ with $0 < \theta < \pi$ also have an absolutely continuous spectrum and $T_{i\pi}$ has only absolutely continuous spectrum. For a more recent discussion about the potential $-x^{-1}$ in the physics literature, we refer to [14, 17, 18, 20], and the references therein.

In § 2 we introduce the symmetric operator S . In § 3 all self-adjoint and maximal dissipative extensions of S are described by an interface condition at 0. Here we use essentially the fact that all these extensions are contained in S^* . There also exist extensions of S in $\mathcal{L}^2([a, b])$ with a non-empty resolvent set which are not contained in S^* [3]. The extensions T_γ , $\gamma \in \mathbb{C} \cup \{\infty\}$, are described in § 4. By a method already used in [12] it is shown that the extensions T_γ for $\gamma \in \mathbb{C}$ can be obtained as norm resolvent limits of operators generated by regular potentials. An analogous result for the case $\gamma = \infty$ can be found in [3]. In § 5 we express the solutions of equation (1.2) by Whittaker functions in order to get information about the characteristic determinant and the asymptotics of the eigenvalues. This is used in § 6, where we prove that the system of root vectors of the operator T_γ forms a Bari basis in $\mathcal{L}^2([a, b])$. Finally, the Fourier coefficients of the corresponding expansions

are expressed by inner products in $\mathcal{L}^2([a, b])$ with the complex conjugate functions of the root functions (which are the root functions of the adjoint operator).

2. The symmetric operator \mathcal{S}

Let $a < 0 < b$. We consider the differential expression $l[f]$ from (1.1) on the intervals $I := [a, b]$, $I_- := [a, 0)$ and $I_+ := (0, b]$; at the endpoints a and b we always impose the Dirichlet boundary conditions (1.3). In the space $\mathcal{L}^2(I_\pm)$ a minimal operator L_\pm and a maximal operator L_\pm^* , which is the adjoint of the minimal operator in $\mathcal{L}^2(I_\pm)$, are associated with the differential expression l . The domain of the maximal operator L_+^* is

$$\mathcal{D}(L_+^*) := \{f \in \mathcal{L}^2(I_+) : f, f' \in \mathcal{AC}_{\text{loc}}(I_+), f(b) = 0, l[f] \in \mathcal{L}^2(I_+)\}$$

and $L_+^*f = l[f]$ if $f \in \mathcal{D}(L_+^*)$. Here, for example, $\mathcal{AC}_{\text{loc}}(I_+)$ is the set of locally absolutely continuous functions on I_+ . The set $\mathcal{D}(L_+^*)$ and the operator L_+^* are defined correspondingly. To describe the domains of the minimal operators L_\pm , we introduce for $f, g \in \mathcal{D}(L_\pm^*)$ and $x, x_1, x_2 \in I_\pm$ the sesquilinear forms

$$[f, g]_x := f(x)\overline{g'(x)} - f'(x)\overline{g(x)}, \quad [f, g]_{x_1}^{x_2} := [f, g]_{x_2} - [f, g]_{x_1}. \quad (2.1)$$

Then Green's formula becomes

$$[f, g]_{x_1}^{x_2} = \int_{x_1}^{x_2} (l[f](x)\overline{g(x)} - f(x)\overline{l[g](x)}) dx. \quad (2.2)$$

It implies that the limits $\lim_{x \rightarrow 0^\pm} [f, g]_x =: [f, g]_{0^\pm}$ exist and are finite and that the sesquilinear forms $[\cdot, \cdot]_{x_1}^{x_2}$ are continuous on $\mathcal{D}(L_\pm^*)$ with respect to the L_\pm^* -graph norms. The domains of the minimal operators can be described as follows [7, theorem 2.3]:

$$\mathcal{D}(L_-) = \{f \in \mathcal{D}(L_-^*) : [f, g]_a^{0-} = 0 \text{ for all } g \in \mathcal{D}(L_-^*)\}, \quad (2.3)$$

$$\mathcal{D}(L_+) = \{f \in \mathcal{D}(L_+^*) : [f, g]_{0+}^b = 0 \text{ for all } g \in \mathcal{D}(L_+^*)\}, \quad (2.4)$$

and Green's formula (2.2) implies that the operators L_\pm are symmetric.

Consider on the interval $[a, b]$ the functions

$$u(x) = x \quad \text{and} \quad v(x) = 1 - x \ln |x|.$$

We choose numbers $\varepsilon_1, \varepsilon_2 : 0 < \varepsilon_1 < \varepsilon_2 < \min\{-a, b\}$ and twice continuously differentiable functions u_\pm on I_\pm with the properties

$$u_+(x) := \begin{cases} u(x) & \text{if } 0 < x < \varepsilon_1, \\ 0 & \text{if } \varepsilon_2 < x < b, \end{cases} \quad u_-(x) := \begin{cases} 0 & \text{if } a < x < -\varepsilon_2, \\ u(x) & \text{if } -\varepsilon_1 < x < 0, \end{cases}$$

and, analogously, functions v_\pm . For x in a neighbourhood of 0,

$$l[u_\pm](x) = -1, \quad l[v_\pm](x) = \ln |x|,$$

hence $l[u_\pm], l[v_\pm] \in \mathcal{L}^2(I_\pm)$ and $u_\pm, v_\pm \in \mathcal{D}(L_\pm^*)$. Further,

$$[v_-, v_-]_a^{0-} = \lim_{x \rightarrow 0-} (v_-(x)\overline{v'_-(x)} - v'_-(x)\overline{v_-(x)}) = 0, \quad (2.5)$$

$$[u_-, v_-]_a^{0-} = \lim_{x \rightarrow 0-} (u_-(x)\overline{v'_-(x)} - v'_-(x)\overline{u_-(x)}) = -1, \quad (2.6)$$

and, analogously,

$$\left. \begin{aligned} [u_-, u_-]_a^{0-} &= [v_+, v_+]_{0+}^b = [u_+, u_+]_{0+}^b = 0, \\ [v_-, u_-]_a^{0-} &= -[v_+, u_+]_{0+}^b = [u_+, v_+]_{0+}^b = 1. \end{aligned} \right\} \quad (2.7)$$

The sesquilinear forms $[\cdot, \cdot]_a^{0-}$ and $[\cdot, \cdot]_{0+}^b$ vanish on $\mathcal{D}(L_-)$ and $\mathcal{D}(L_+)$, respectively; see equations (2.3) and (2.4). Therefore, the functions u_{\pm} and v_{\pm} are linearly independent modulo $\mathcal{D}(L_{\pm})$. Since l is a second-order differential operator and boundary conditions at a and b have been fixed, the dimension of the factor space $\mathcal{D}(L_{\pm}^*)/\mathcal{D}(L_{\pm})$ is at most 2, and we find

$$\mathcal{D}(L_-^*) = \mathcal{D}(L_-) \dot{+} \text{span}\{u_-, v_-\}, \quad \mathcal{D}(L_+^*) = \mathcal{D}(L_+) \dot{+} \text{span}\{u_+, v_+\}. \quad (2.8)$$

Now we consider in the Hilbert space

$$\mathcal{L}^2(I) = \mathcal{L}^2(I_-) \oplus \mathcal{L}^2(I_+) \quad (2.9)$$

the operator $S := L_- \oplus L_+$. Evidently, $S^* = L_-^* \oplus L_+^*$ and on $\mathcal{D}(S^*)$ we define the sesquilinear form

$$[f, g] := [f_-, g_-]_a^{0-} + [f_+, g_+]_{0+}^b, \quad f, g \in \mathcal{D}(S^*), \quad (2.10)$$

where $f = f_- + f_+$ and $g = g_- + g_+$ are the decompositions of the elements f and g with respect to (2.9). Relation (2.2) implies the Green's formula

$$[f, g] = (S^* f, g) - (f, S^* g), \quad f, g \in \mathcal{D}(S^*), \quad (2.11)$$

and the sesquilinear form on the left-hand side is again continuous in the S^* -graph norm on $\mathcal{D}(S^*)$.

We extend the functions u_{\pm} and v_{\pm} to the whole interval $[a, b]$ as follows:

$$\tilde{u}_-(x) := \begin{cases} u_-(x) & \text{if } x \in [a, 0), \\ 0 & \text{if } x \in (0, b], \end{cases} \quad \tilde{u}_+(x) := \begin{cases} 0 & \text{if } x \in [a, 0), \\ u_+(x) & \text{if } x \in (0, b], \end{cases}$$

and \tilde{v}_{\pm} are defined analogously. All these extended functions belong to $\mathcal{D}(S^*)$. On $f \in \mathcal{D}(S^*)$ the following functionals u_{\pm}^* , v_{\pm}^* are defined:

$$u_-^* f := [f, \tilde{u}_-], \quad u_+^* f := [f, \tilde{u}_+], \quad v_-^* f := [f, \tilde{v}_-], \quad v_+^* f := [f, \tilde{v}_+]. \quad (2.12)$$

From (2.3) and (2.4) it follows that the functionals u_{\pm}^* , v_{\pm}^* vanish on $\mathcal{D}(S)$, and the definition of the functions \tilde{u}_{\pm} , \tilde{v}_{\pm} yields for $f \in \mathcal{D}(S^*)$ the relations

$$u_{\pm}^* f = \mp f(0_{\pm}), \quad v_{\pm}^* f = \pm \lim_{x \rightarrow 0_{\pm}} (f'(x) + f(x)(1 + \ln|x|)), \quad (2.13)$$

where we have used that the functions $f \in \mathcal{D}(S^*)$ satisfy the relation

$$f'(x) = O(\ln|x|) \quad \text{for } x \rightarrow 0; \quad (2.14)$$

see [8, lemma 2.2]. Since the operators L_{\pm} are symmetric, also S is a symmetric operator and we have

$$\mathcal{D}(S) = \{f \in \mathcal{D}(S^*): u_-^* f = u_+^* f = v_-^* f = v_+^* f = 0\} \quad (2.15)$$

and

$$\mathcal{D}(S^*) = \mathcal{D}(S) + \text{span}\{\tilde{u}_-, \tilde{u}_+, \tilde{v}_-, \tilde{v}_+\}. \quad (2.16)$$

Therefore, the defect index of the operator S is $(2, 2)$.

LEMMA 2.1. *If $f \in \mathcal{D}(S)$, it holds that*

$$f(x) = o(x), \quad f'(x) = o(1) \quad \text{for } x \rightarrow 0, \quad (2.17)$$

and

$$\mathcal{D}(S) = \{f \in \mathcal{D}(S^*) : f, f' \text{ are continuous in } 0 \text{ and } f(0) = f'(0) = 0\}. \quad (2.18)$$

Proof. If $f \in \mathcal{D}(S)$, then (2.15) and the first relation in (2.13) imply, for $x \rightarrow 0$,

$$f(x) = o(1). \quad (2.19)$$

Now relation (2.14) yields the sharper estimate

$$f(x) = \int_0^x f'(t) dt = O(x \ln |x|), \quad (2.20)$$

and if we observe that ${}^{\mathbf{v}}_{\pm} f = 0$, it follows by (2.15) and the second relation in (2.13) that

$$f'(x) = -(1 + \ln |x|)O(x \ln |x|) + o(1) = o(1)$$

and finally

$$f(x) = \int_0^x f'(t) dt = o(x).$$

Thus the relations (2.17) and the inclusion

$$\mathcal{D}(S) \subset \{f \in \mathcal{D}(S^*) : f, f' \text{ are continuous in } 0 \text{ and } f(0) = f'(0) = 0\}$$

are proved. The equality sign in (2.18) follows now from (2.16) and the fact that no linear combination f of the functions u_{\pm}, v_{\pm} , except the trivial one, has the property that f and f' are continuous and fulfil $f(0) = f'(0) = 0$. \square

3. The self-adjoint and the maximal dissipative extensions of S

The symmetric operator S in $\mathcal{L}^2(I)$ with defect index $(2, 2)$, which was associated with the differential expression l from (1.1) and the Dirichlet boundary conditions (1.3), has self-adjoint and maximal dissipative canonical extensions; here *canonical* means that these extensions act in the originally given space $\mathcal{L}^2(I)$. We shall characterize these extensions by interface conditions at 0.

To this end, we first observe that all symmetric and dissipative canonical extensions of S are restrictions of the adjoint S^* (see [11, theorem 3.1.3] and [15, theorem 1.3.7]). Relation (2.15) implies that such an extension is determined by a linear relation between the functionals ${}^{\mathbf{u}}_{\pm}, {}^{\mathbf{v}}_{\pm}$, which are defined on $\mathcal{D}(S^*)$. Let $b: \mathcal{D}(S^*) \rightarrow \mathbb{C}^4$ be the mapping

$$b := \begin{pmatrix} u_- & v_- & u_+ & v_+ \end{pmatrix}^T, \quad (3.1)$$

by J_0 we denote the 2×2 matrix

$$J_0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (3.2)$$

and by J the 4×4 matrix

$$J = \begin{pmatrix} J_0 & 0 \\ 0 & -J_0 \end{pmatrix}.$$

PROPOSITION 3.1. *The linear mapping $\overset{b}{\cdot}$ from (3.1) has these properties:*

- (i) $\mathcal{R}(\overset{b}{\cdot}) = \mathbb{C}^4$,
- (ii) $\ker \overset{b}{\cdot} = \mathcal{D}(S)$,
- (iii) $\frac{(S^*f, g) - (f, S^*g)}{i} = (\overset{b}{g})^* J \overset{b}{f}, \quad f, g \in \mathcal{D}(S^*)$.

Proof. The definitions (2.12) and the relations (2.5), (2.6) and (2.7) imply

$$\overset{b}{\tilde{u}}_- = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \overset{b}{\tilde{v}}_- = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \overset{b}{\tilde{u}}_+ = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \overset{b}{\tilde{v}}_+ = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad (3.3)$$

and (i) follows. Statement (ii) is a consequence of (2.15).

In order to prove (iii), we observe that, according to (2.16), each $f \in \mathcal{D}(S^*)$ is a linear combination of an element $f_0 \in \mathcal{D}(S)$ and $\tilde{u}_\pm, \tilde{v}_\pm$. Relations (2.5), (2.6) and (2.7) imply that $f = f_0 + f_1$ with $f_0 \in \mathcal{D}(S)$ and

$$f_1 := (\overset{u}{-}f)\tilde{v}_- - (\overset{v}{-}f)\tilde{u}_- - (\overset{u}{+}f)\tilde{v}_+ + (\overset{v}{+}f)\tilde{u}_+.$$

With an analogous decomposition of $g \in \mathcal{D}(S^*)$ it follows from (2.11), (2.3) and (2.4) that

$$\frac{(S^*f, g) - (f, S^*g)}{i} = \frac{[f_1, g_1]}{i}.$$

By means of (2.11), (2.5), (2.6) and (2.7) we find for the expression on the right-hand side the form

$$(\overset{b}{g})^* J \overset{b}{f},$$

and relation (iii) is proved. □

We equip the space \mathbb{C}^4 with the inner product generated by J : $(Jx, y) := y^* Jx$. Then a subspace \mathcal{U} of \mathbb{C}^4 is called *J-non-negative* (*J-neutral*, respectively) if $(Jx, x) \geq 0$ ($= 0$, respectively) for all $x \in \mathcal{U}$.

COROLLARY 3.2. *The operator T is a (maximal) dissipative canonical extension of S if and only if $\mathcal{U} = \{\overset{b}{f} : f \in \mathcal{D}(T)\}$ is a (maximal) J-non-negative subspaces of \mathbb{C}^4 , and T is a (maximal) symmetric canonical extension of S if and only if this subspace is (maximal) J-neutral.*

Indeed, it follows from statement (iii) of proposition 3.1 that the operator $T \subset S^*$ is, for example, dissipative if and only if, for all $f \in \mathcal{D}(T)$, it holds that

$$0 \leq 2 \operatorname{Im}(Tf, f) = \frac{(Tf, f) - (f, Tf)}{i} = \frac{(S^*f, f) - (f, S^*f)}{i} = (b_f)^* J b_f.$$

The other claims follow in the same way.

In the sequel, B denotes a complex 2×4 matrix, which we write also as a block matrix

$$B = \begin{pmatrix} C & D \end{pmatrix}$$

with two 2×2 matrices C and D ; J_0 is the matrix defined in (3.2). Since the eigenvalues of the matrix J are ± 1 , each of multiplicity 2, the maximal J -non-negative subspaces of \mathbb{C}^4 are of dimension 2.

THEOREM 3.3. *The operator T is a maximal dissipative canonical extension of S if and only if*

$$\mathcal{D}(T) = \{f \in \mathcal{D}(S^*) : B^b f = 0\}, \quad (3.4)$$

where the 2×4 matrix $B = \begin{pmatrix} C & D \end{pmatrix}$ is such that its rank is 2 and the inequality

$$C J_0 C^* \leq D J_0 D^* \quad (3.5)$$

holds; T is a self-adjoint canonical extension of S if and only if the rank of the matrix B in (3.4) is 2 and the relation

$$C J_0 C^* = D J_0 D^* \quad (3.6)$$

holds.

Proof. By corollary 3.2, T is maximal dissipative if and only if $\mathcal{U} = \{b_f : f \in \mathcal{D}(T)\} = \ker B$ is maximal J -non-negative. This is the case if and only if $\mathcal{U}^\perp = \mathcal{R}(B^*)$ is maximal J -non-positive, which is equivalent to (3.5) and $\operatorname{rank} B = 2$. The proof of the second statement of the theorem is similar. \square

If we write the matrices C and D in the form

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \quad D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix},$$

the interface condition $B^b f = 0$ in (3.4) becomes

$$\begin{aligned} c_{11}f(0-) - c_{12} \lim_{x \rightarrow 0-} (f'(x) + (1 + \ln|x|)f(x)) \\ - d_{11}f(0+) + d_{12} \lim_{x \rightarrow 0+} (f'(x) + (1 + \ln|x|)f(x)) &= 0, \\ c_{21}f(0-) - c_{22} \lim_{x \rightarrow 0-} (f'(x) + (1 + \ln|x|)f(x)) \\ - d_{21}f(0+) + d_{22} \lim_{x \rightarrow 0+} (f'(x) + (1 + \ln|x|)f(x)) &= 0. \end{aligned} \quad (3.7)$$

4. Continuity at the origin

In this section we consider those maximal dissipative canonical extensions T of the symmetric operator S for which the functions $f \in \mathcal{D}(T)$ are continuous at zero. Continuity of f at zero means that $f(0-) = f(0+)$, which according to (2.13) is equivalent to $\overset{u}{-}f + \overset{u}{+}f = 0$. Therefore, these extensions are described by a matrix B with the property

$$c_{11} = d_{11} \neq 0, \quad c_{12} = d_{12} = 0,$$

and we can assume that

$$C = \begin{pmatrix} 1 & 0 \\ c_{21} & c_{22} \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ d_{21} & d_{22} \end{pmatrix}.$$

Condition (3.5) is equivalent to

$$c_{22} = d_{22} \quad \text{and} \quad \frac{c_{21}\overline{c_{22}} - c_{22}\overline{c_{21}}}{i} \leq \frac{d_{21}\overline{d_{22}} - d_{22}\overline{d_{21}}}{i}. \quad (4.1)$$

If $c_{22} = d_{22} = 0$, matrix B can be supposed to have the form

$$B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

If $c_{22} = d_{22} \neq 0$ we can assume that this number is 1, and inequality (4.1) becomes $\operatorname{Im} c_{21} \leq \operatorname{Im} d_{21}$. By subtracting a multiple of the first row of B from the second row, we arrive at the following result.

THEOREM 4.1. *The functions in the domain of the maximal dissipative canonical extension T of S are continuous in 0 if and only if the matrix B in (3.4) can be chosen as*

$$B_\gamma = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & \gamma & 1 \end{pmatrix} \quad \text{with} \quad \operatorname{Im} \gamma \geq 0, \quad (4.2)$$

or as

$$B_\infty = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (4.3)$$

This extension T is self-adjoint if and only if in (4.2) $\operatorname{Im} \gamma = 0$ or if B is of the form (4.3).

The extension T of S having the form (3.4) with $B = B_\gamma$ is denoted by T_γ , $\gamma \in \mathbb{C}^+ \cup \{\infty\}$. It is easy to see that also for $\gamma \in \mathbb{C}^-$ an extension T_γ is defined by the same interface conditions; then the operator $-T_\gamma$ is maximal dissipative.

In order to write the boundary conditions for the extension T_γ in a more explicit form than (3.7), we need a lemma.

LEMMA 4.2. *If $f \in \mathcal{D}(S^*)$ and $f(0+) = f(0-)$, then*

$$\lim_{x \rightarrow 0+} (f(x) - f(-x))(1 + \ln |x|) = 0. \quad (4.4)$$

Proof. If $f \in \mathcal{D}(S)$, the claim follows from (2.17). So it remains to consider linear combinations

$$f = \alpha_- \tilde{u}_- + \beta_- \tilde{v}_- + \alpha_+ \tilde{u}_+ + \beta_+ \tilde{v}_+,$$

for which, because of the continuity of f at 0, also $\beta_- = \beta_+ =: \beta$. Hence f has the form

$$f = \alpha_- \tilde{u}_- + \alpha_+ \tilde{u}_+ + \beta v,$$

and relation (4.4) follows easily from the definition of functions \tilde{u}_\pm and v . \square

THEOREM 4.3. *The extension T_γ , $\gamma \in \mathbb{C} \cup \{\infty\}$, of S is given by interface conditions of the form*

$$f(0-) = f(0+), \quad \lim_{x \rightarrow 0+} (f'(x) - f'(-x)) = \gamma f(0) \quad \text{if } \gamma \in \mathbb{C}, \quad (4.5)$$

and by the Dirichlet interface conditions

$$f(0+) = f(0-) = 0 \quad \text{if } \gamma = \infty. \quad (4.6)$$

T_γ is self-adjoint if and only if $\gamma \in \mathbb{R} \cup \{\infty\}$.

Proof. If the matrix $B = B_\gamma$ given by (4.2), then the interface conditions at 0 for $f \in \mathcal{D}(T) \subset \mathcal{D}(S^*)$ are $f(0-) = f(0+)$ and

$$-\lim_{x \rightarrow 0-} (f'(x) + (1 + \ln|x|)f(x)) + \lim_{x \rightarrow 0+} (f'(x) + (1 + \ln|x|)f(x)) = \gamma f(0+). \quad (4.7)$$

By lemma 4.2, relation (4.7) is equivalent to relation (4.5). If the matrix $B = B_\infty$ given by (4.3), we obtain the Dirichlet interface conditions. \square

For the canonical extensions of S which were considered in [12], it was shown there that they are norm resolvent limits of Sturm–Liouville operators with a regular potential. We show by the same method as in [12] that this is true for all the operators T_γ , $\gamma \in \mathbb{C}$. To this end, we define for $\gamma \in \mathbb{C}$ and $\varepsilon > 0$ the Sturm–Liouville operators $T_{\gamma,\varepsilon}$ as follows:

$$\begin{aligned} \mathcal{D}(T_{\gamma,\varepsilon}) &:= \{f \in \mathcal{L}^2(I) : f, f' \in \mathcal{AC}_{\text{loc}}(I), f'' \in \mathcal{L}^2(I), f(a) = f(b) = 0\}, \\ (T_{\gamma,\varepsilon}f)(x) &:= -f''(x) - \frac{1}{2} \left(\frac{1 + \gamma/i\pi}{x + i\varepsilon} + \frac{1 - \gamma/i\pi}{x - i\varepsilon} \right) f(x). \end{aligned}$$

THEOREM 4.4. *For $\gamma \in \mathbb{C}$, the operator T_γ is the norm resolvent limit of the operators $T_{\gamma,\varepsilon}$ if $\varepsilon \rightarrow 0+$.*

Proof. On the set

$$\mathcal{D} := \{f \in \mathcal{AC}_{\text{loc}}(I) : f' \in \mathcal{L}^2(I), f(a) = f(b) = 0\}$$

we consider the following sesquilinear forms:

$$\begin{aligned}\mathfrak{l}^0[f, g] &:= \int_a^b f'(x) \overline{g'(x)} \, dx, \\ \mathfrak{q}_\varepsilon[f, g] &:= - \int_a^b \frac{f(x) \overline{g(x)}}{x + i\varepsilon} \, dx, \text{ if } \varepsilon \neq 0, \quad \mathfrak{q}_0[f, g] := -P \int_a^b \frac{f(x) \overline{g(x)}}{x} \, dx, \\ \mathfrak{b}[f, g] &:= f(0) \overline{g(0)},\end{aligned}$$

where P denotes the Cauchy principal value. The form \mathfrak{l}^0 is closed and non-negative; the forms \mathfrak{q}_0 and \mathfrak{b} are symmetric and \mathfrak{l}^0 -bounded with relative bound zero [12, lemmas 2.3 and 2.5]. Hence, according to [13, theorem VI.1.33],

$$\mathfrak{t}_\gamma := \mathfrak{l}^0 + \mathfrak{q}_0 + \gamma \mathfrak{b}$$

is a closed sectorial form on \mathcal{D} . By the second representation theorem [13, theorem VI.2.1], there exists an m -sectorial operator T_{t_γ} such that

1. $\mathcal{D}(T_{t_\gamma}) \subset \mathcal{D}$;
2. $\mathfrak{t}_\gamma[f, g] = (T_{t_\gamma} f, g)$, $f \in \mathcal{D}(T_{t_\gamma})$, $g \in \mathcal{D}$;
3. $\mathcal{D}(T_{t_\gamma})$ is a core of \mathfrak{t}_γ ;
4. if $f \in \mathcal{D}$, $y \in \mathcal{L}^2(I)$ such that the equality $\mathfrak{t}_\gamma[f, g] = (y, g)$ holds for all g in a core of \mathfrak{t}_γ , then $f \in \mathcal{D}(T_{t_\gamma})$ and $T_{t_\gamma} f = y$.

We shall show that $T_{t_\gamma} = T_\gamma$. Theorem 4.3 implies $\mathcal{D}(T_\gamma) \subset \mathcal{D}$, and for $f \in \mathcal{D}(T_\gamma)$ and $g \in \mathcal{D}$ it holds that

$$\begin{aligned}(T_\gamma f, g) &= \left(\int_a^0 + \int_0^b \right) \left(-f''(x) - \frac{f(x)}{x} \right) \overline{g(x)} \, dx \\ &= \lim_{\varepsilon \rightarrow 0+} \left(\int_a^{-\varepsilon} + \int_\varepsilon^b \right) \left(-f''(x) - \frac{f(x)}{x} \right) \overline{g(x)} \, dx \\ &= P \int_a^b \left(f'(x) \overline{g'(x)} - \frac{f(x) \overline{g(x)}}{x} \right) \, dx + \lim_{\varepsilon \rightarrow 0+} (f'(\varepsilon) \overline{g(\varepsilon)} - f'(-\varepsilon) \overline{g(-\varepsilon)}) \\ &= \mathfrak{l}^0[f, g] + \mathfrak{q}_0[f, g] + \lim_{\varepsilon \rightarrow 0+} (f'(\varepsilon) \overline{g(\varepsilon)} - f'(-\varepsilon) \overline{g(-\varepsilon)}) \\ &= \mathfrak{t}_\gamma[f, g] + \lim_{\varepsilon \rightarrow 0+} (f'(\varepsilon) \overline{g(\varepsilon)} - f'(-\varepsilon) \overline{g(-\varepsilon)}) - \gamma f(0) \overline{g(0)}.\end{aligned}\tag{4.8}$$

If $g \in \mathcal{D}$, we have

$$|g(x) - g(0)| \leq \int_0^x |g'(s)| \, ds \leq \sqrt{|x|} \|g'(s)\|.$$

Therefore, relation (2.14) yields, for $f \in \mathcal{D}(T_\gamma)$,

$$\begin{aligned}\lim_{x \rightarrow 0+} (f'(x) \overline{g(x)} - f'(-x) \overline{g(-x)}) - \gamma f(0) \overline{g(0)} \\ = \lim_{x \rightarrow 0+} (f'(x) - f'(-x) - \gamma f(0)) \overline{g(0)} = 0.\end{aligned}$$

Hence (4.8) becomes

$$(T_\gamma f, g) = t_\gamma[f, g], \quad f \in \mathcal{D}(T_\gamma), \quad g \in \mathcal{D},$$

which implies $T_\gamma \subset T_{t_\gamma}$. Since, on the other hand, T_γ or $-T_\gamma$ is a maximal dissipative operator, in this inclusion the equality sign must prevail.

The differential operator $T_{\gamma, \varepsilon}$ is associated with the sesquilinear form

$$t_{\gamma, \varepsilon} = l^0 + \frac{\pi i + \gamma}{2\pi i} q_\varepsilon + \frac{\pi i - \gamma}{2\pi i} q_{-\varepsilon},$$

which is also defined on \mathcal{D} . As in the proof of [12, theorem 3.3], for $f, g \in \mathcal{D}$ it follows that

$$|q_{\pm\varepsilon}[f, g] - (q_0[f, g] \mp \pi i b[f, g])| = o(1)l^0[f, g] + o(1)(f, g), \quad \varepsilon \rightarrow 0+,$$

and we get

$$t_{\gamma, \varepsilon}[f, g] - t_\gamma[f, g] = o(1)l^0[f, g] + o(1)(f, g), \quad \varepsilon \rightarrow 0+.$$

Now the resolvent convergence of the operators $T_{\gamma, \varepsilon}$ to T_γ follows from [13, theorem VI.3.6]. \square

5. Representation of the solutions by Whittaker functions

In this section we express the resolvents of the extensions T_γ from § 4 by means of Whittaker functions. We first recall Whittaker's differential equation [2, 5, 16, 21]:

$$\frac{d^2 f(z)}{dz^2} + \left(-\frac{1}{4} + \frac{\kappa}{z} + \frac{1 - \mu^2}{4z^2} \right) f(z) = 0. \quad (5.1)$$

Two linearly independent solutions of this differential equation are the Whittaker functions

$$\begin{aligned} M_{\kappa, \mu(z)/2} &= z^{(1+\mu)/2} e^{-z/2} \Phi\left(\frac{1}{2}(1+\mu) - \kappa, 1 + \mu, z\right), \\ W_{\kappa, \mu(z)/2} &= z^{(1+\mu)/2} e^{-z/2} \Psi\left(\frac{1}{2}(1+\mu) - \kappa, 1 + \mu, z\right), \end{aligned}$$

where Φ is the confluent hypergeometric function. In the following we use the function $\psi(z) := \Gamma'(z)/\Gamma(z)$, and for complex numbers α and β and an integer k the symbols

$$(\alpha)_k := \alpha(\alpha+1) \cdots (\alpha+k-1), \quad d_k(\alpha, \beta) := \psi(\alpha+k) - \psi(1+k) - \psi(\beta+k).$$

Then the function Φ is given by the relation

$$\Phi(\alpha, \beta, z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\beta)_k} \frac{z^k}{k!},$$

and in the case that β is a positive integer, $\Psi(\alpha, \beta, z)$ admits the following representation [2, § 6.1, § 6.7, formula (13)]:

$$\begin{aligned} \Psi(\alpha, \beta, z) = & \frac{(-1)^\beta}{\Gamma(\beta)\Gamma(\alpha - \beta + 1)} \left(\Phi(\alpha, \beta, z) \ln z + \sum_{k=0}^{\infty} \frac{(\alpha)_k d_k(\alpha, \beta) z^k}{(\beta)_k k!} \right) \\ & + \frac{\Gamma(\beta - 1)}{\Gamma(\alpha)} \sum_{k=0}^{\beta-2} \frac{(\alpha - \beta + 1)_k z^{k-\beta+1}}{(2 - \beta)_k k!}. \end{aligned} \quad (5.2)$$

If we make the substitution

$$\mu = 1, \quad \kappa = \frac{i}{2\sqrt{\lambda}}, \quad z = \frac{x}{\kappa} = -2i\sqrt{\lambda}x,$$

equation (5.1) becomes equation (1.2): $l[f] - \lambda f = 0$. Therefore, two linearly independent solutions of (1.2) are the functions

$$\left. \begin{aligned} f_M(x, \lambda) &= M_{i/2\sqrt{\lambda}, 1/2}(-2i\sqrt{\lambda}x), \\ f_W(x, \lambda) &= \Gamma(1 - i/2\sqrt{\lambda}) W_{i/2\sqrt{\lambda}, 1/2}(-2i\sqrt{\lambda}x); \end{aligned} \right\} \quad (5.3)$$

see also [4, 8]. The function f_M is entire in x , whereas f_W has a logarithmic branch point at $x = 0$. The function f_W is understood as the principal branch, which is obtained from the principal branch of the logarithm in (5.2).

With the functions $f_M(x, \lambda)$ and $f_W(x, \lambda)$ we form for $\lambda \neq 0$ the solutions

$$f_-(x, \lambda) := \begin{cases} \frac{f_M(a, \lambda)f_W(x, \lambda) - f_W(a, \lambda)f_M(x, \lambda)}{f_M(a, \lambda)f'_W(a, \lambda) - f_W(a, \lambda)f'_M(a, \lambda)} & \text{if } x < 0, \\ 0 & \text{if } x > 0, \end{cases} \quad (5.4)$$

$$f_+(x, \lambda) := \begin{cases} 0 & \text{if } x < 0, \\ \frac{f_M(b, \lambda)f_W(x, \lambda) - f_W(b, \lambda)f_M(x, \lambda)}{f_M(b, \lambda)f'_W(b, \lambda) - f_W(b, \lambda)f'_M(b, \lambda)} & \text{if } x > 0. \end{cases} \quad (5.5)$$

They satisfy for $x \neq 0$ the differential equation $l[f] - \lambda f = 0$ and the boundary conditions

$$\begin{aligned} f_-(a, \lambda) &= 0, & f'_-(a, \lambda) &= 1, \\ f_+(b, \lambda) &= 0, & f'_+(b, \lambda) &= 1. \end{aligned}$$

If $x \neq 0$ is fixed, $f_\pm(x, \lambda)$ are entire functions in λ . Further, we introduce the kernel

$$K(x, \xi; \lambda) := \begin{cases} \frac{f_M(\xi, \lambda)f_W(x, \lambda) - f_W(\xi, \lambda)f_M(x, \lambda)}{f_M(\xi, \lambda)f'_W(\xi, \lambda) - f_W(\xi, \lambda)f'_M(\xi, \lambda)} & \text{if } \xi \leq x < 0, \\ -\frac{f_M(\xi, \lambda)f_W(x, \lambda) - f_W(\xi, \lambda)f_M(x, \lambda)}{f_M(\xi, \lambda)f'_W(\xi, \lambda) - f_W(\xi, \lambda)f'_M(\xi, \lambda)} & \text{if } 0 < x \leq \xi, \\ 0 & \text{otherwise.} \end{cases}$$

It satisfies for $x \neq 0$ and $x \neq \xi$ the differential equation

$$-\frac{\partial^2 K}{\partial x^2}(x, \xi; \lambda) - \frac{K(x, \xi; \lambda)}{x} = \lambda K(x, \xi; \lambda)$$

and the boundary conditions

$$\frac{\partial K}{\partial x}(\xi+, \xi; \lambda) = 1 \text{ if } \xi < 0, \quad \frac{\partial K}{\partial x}(\xi-, \xi; \lambda) = -1 \text{ if } \xi > 0.$$

We introduce the following operators K_λ , $\lambda \in \mathbb{C}$, in $\mathcal{L}^2(I)$:

$$(K_\lambda f)(x) := \int_a^b K(x, \xi; \lambda) f(\xi) d\xi, \quad f \in \mathcal{L}^2(I).$$

Then $K_\lambda f \in \mathcal{D}(S^*)$ and $(S^* - \lambda)K_\lambda f = f$ for arbitrary $f \in \mathcal{L}^2(I)$. This implies for functions $f \in \mathcal{D}(S^*)$ that $K_\lambda(S^* - \lambda)f = f + g$ with $g \in \ker(S^* - \lambda)$. If f vanishes identically near a and b , then also $K_\lambda(S^* - \lambda)f$ does. In this case $g = 0$, and $K_\lambda(S^* - \lambda)f = f$, which yields $\tilde{u}_\pm, \tilde{v}_\pm \in \mathcal{R}(K_\lambda)$ and further

$$\mathcal{R}(^b K_\lambda) = \mathbb{C}^4. \quad (5.6)$$

The functions $f_-(\cdot, \lambda)$ and $f_+(\cdot, \lambda)$ span the kernel $\ker(S^* - \lambda)$. For given $f \in \mathcal{L}^2(I)$ the equation

$$(T_\gamma - \lambda)f = y \quad (5.7)$$

is satisfied if and only if $f = c_- f_- + c_+ f_+ + K_\lambda y$ with numbers c_- and c_+ such that $B_\gamma (^b c_- f_- + c_+ f_+ + K_\lambda y) = 0$. Relation (5.6) implies that the latter equation has a unique solution for arbitrary $y \in \mathcal{L}^2(I)$ if and only if the 2×2 matrix

$$M_\gamma(\lambda) := (B_\gamma (^b f_-(\cdot; \lambda) \quad B_\gamma (^b f_+(\cdot; \lambda)) \quad (5.8)$$

is invertible, and the solution of equation (5.7) can be written as

$$f(x) = (K_\lambda y)(x) - (f_-(x, \lambda) \quad f_+(x, \lambda)) M_\gamma(\lambda)^{-1} B_\gamma (^b (K_\lambda y)). \quad (5.9)$$

For the following theorem see [19, I § 2].

THEOREM 5.1. *Suppose $\gamma \in \mathbb{C}$ and let $M_\gamma(\lambda)$ be the matrix function from (5.8). Then $\lambda \in \rho(T_\gamma)$ if and only if $\det M_\gamma(\lambda) \neq 0$, and in this case the resolvent $(T_\gamma - \lambda)^{-1}$ is given by (5.9): $(T_\gamma - \lambda)^{-1}y = f$. The eigenvalues of T_γ are geometrically simple, and the length of the Jordan chain of T_γ at an eigenvalue λ equals the order of the zero $\zeta = \lambda$ of the function $\det M_\gamma(\zeta)$.*

Proof. If $\det M_\gamma(\lambda) \neq 0$, the resolvent $(T_\gamma - \lambda)^{-1}$ exists and is given by (5.9). Now suppose that $\det M_\gamma(\lambda) = 0$. Then the non-zero 2-vector $(c_-, c_+)^T$ belongs to $\ker M_\gamma(\lambda)$ if and only if the function $f(x) := c_- f_-(x, \lambda) + c_+ f_+(x, \lambda)$ fulfils the interface condition $B_\gamma (^b f = 0$ and hence is an eigenfunction of T_γ at λ . Since all eigenfunctions of T_γ at λ are of this form and the matrix $M_\gamma(\lambda)$ is not the zero matrix, the geometric multiplicity of the eigenvalue λ equals one.

Suppose now that λ is a zero of order m of the function $\det M_\gamma(\zeta)$. Then (5.9) implies that the length of the Jordan chain of T_γ at λ is at most m . A chain of length m can be obtained as follows. Since

$$M_\gamma(\zeta) = \begin{pmatrix} m_{\gamma,11}(\zeta) & m_{\gamma,12}(\zeta) \\ m_{\gamma,21}(\zeta) & m_{\gamma,22}(\zeta) \end{pmatrix}$$

is not the zero matrix, at least one entry does not vanish. Suppose, for example, that this is $m_{\gamma,11}(\lambda)$; the other cases can be treated similarly. With the matrices

$$E(\zeta) = \begin{pmatrix} 1 & 0 \\ -\frac{m_{\gamma,21}(\zeta)}{m_{\gamma,11}(\zeta)} & 1 \end{pmatrix}, \quad F(\zeta) = \begin{pmatrix} 1 & -\frac{m_{\gamma,12}(\zeta)}{m_{\gamma,11}(\zeta)} \\ 0 & 1 \end{pmatrix}$$

we get

$$E(\zeta)M_\gamma(\zeta)F(\zeta) = \begin{pmatrix} m_{\gamma,11}(\zeta) & 0 \\ 0 & \frac{\det M_\gamma(\zeta)}{m_{\gamma,11}(\zeta)} \end{pmatrix}.$$

Therefore, the analytic family of vectors

$$\begin{pmatrix} c_-(\zeta) \\ c_+(\zeta) \end{pmatrix} = F(\zeta) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

fulfils for $\zeta \rightarrow \lambda$ the relations

$$\begin{pmatrix} c_-(\zeta) \\ c_+(\zeta) \end{pmatrix} \not\rightarrow 0, \quad \begin{pmatrix} d_-(\zeta) \\ d_+(\zeta) \end{pmatrix} = M_\gamma(\zeta) \begin{pmatrix} c_-(\zeta) \\ c_+(\zeta) \end{pmatrix} = O((\zeta - \lambda)^m).$$

Then (3.3) and (5.8) give

$$f(\cdot, \zeta) = c_-(\zeta)f_-(\cdot, \zeta) + c_+(\zeta)f_+(\cdot, \zeta) - d_-(\zeta)\tilde{v}_-(\cdot) - d_+(\zeta)\tilde{u}_+(\cdot) \in \mathcal{D}(T_\gamma),$$

and the relation $(T_\gamma - \zeta)f(\cdot, \zeta) = O((\zeta - \lambda)^m)$ implies that the functions

$$f_i(\cdot, \lambda) := \frac{\partial f(\cdot, \lambda)}{\partial \lambda^i}, \quad i = 0, 1, \dots, m-1,$$

form a Jordan chain at λ . □

In the following we need some asymptotic properties of the eigenvalues of the operators T_γ . To this end, we study the asymptotic behaviour of the functions f_M and f_W . The relations (5.3) imply the following asymptotics. If $\lambda \in \mathbb{C} \setminus \{0\}$ is fixed, then for $x \rightarrow 0$,

$$f_M(x, \lambda) = -2i\sqrt{\lambda}x + O(x^2), \quad (5.10)$$

$$\begin{aligned} f_W(x, \lambda) &= e^{-z/2} - \kappa z e^{-z/2} ((1 + O(z) \ln z + d_0(1 - \kappa, 2) + O(z)) \\ &= 1 + i\sqrt{\lambda}x - \ln z - d_0(1 - \kappa, 2)x + O(x^2 \ln x) \\ &= 1 - x \ln |x| + c_\lambda(x)x + O(x^2 \ln x), \end{aligned} \quad (5.11)$$

where

$$c_\lambda(x) := i\sqrt{\lambda} - d_0\left(1 - \frac{i}{2\sqrt{\lambda}}, 2\right) + \ln |x| - \ln(-2i\sqrt{\lambda}x). \quad (5.12)$$

Note that $c_\lambda(x)$ does not depend on $|x|$, hence it is bounded if $x \rightarrow \pm 0$. Further, it holds that

$$c_\lambda(+1) - c_\lambda(-1) = \ln(2i\sqrt{\lambda}) - \ln(-2i\sqrt{\lambda}) = i\pi. \quad (5.13)$$

Relations (5.10) and (5.11) imply

$$f_W(0-, \lambda) = f_W(0+, \lambda) = 1, \quad f_M(0-, \lambda) = f_M(0+, \lambda) = 0 \quad (5.14)$$

and

$$\left. \begin{aligned} \lim_{x \rightarrow 0} (f'_M(x, \lambda) + (1 + \ln |x|)f_M(x, \lambda)) &= -2i\sqrt{\lambda}, \\ \lim_{x \rightarrow 0-} (f'_W(x, \lambda) + (1 + \ln |x|)f_W(x, \lambda)) &= c_\lambda(-1), \\ \lim_{x \rightarrow 0+} (f'_W(x, \lambda) + (1 + \ln |x|)f_W(x, \lambda)) &= c_\lambda(+1), \end{aligned} \right\} \quad (5.15)$$

where $c_\lambda(x)$ is given by (5.12).

REMARK 5.2. Boyd [4] considered the boundary value problem (1.1) with boundary conditions (1.3), replacing the potential $-x^{-1}$ first by $-(x - i\varepsilon)^{-1}$ with $\varepsilon > 0$ and letting $\varepsilon \rightarrow 0$. He required the eigenfunctions to admit an analytic continuation onto the lower half-plane. This requirement specifies an interface condition in $x = 0$, which, however, turns out not to be self-adjoint. Indeed, the solutions of (1.1) which admit an analytic continuation onto the lower half-plane are linear combinations of the functions $f_M(x, \lambda)$ and $\tilde{f}_W(x, \lambda)$, where $\tilde{f}_W(x, \lambda)$ equals the function $f_W(x, \lambda)$ for positive real x , and with the branch cut at $\arg x = \pi/2$. This corresponds to a branch cut in the logarithm in the definition of the function Ψ in (5.2) at $\arg z = \arg \sqrt{\lambda}$. For real x and $-\pi < \arg \lambda \leq \pi$, this means

$$\tilde{f}_W(x, \lambda) = \begin{cases} f_W(x, \lambda) & \text{if } x > 0, \\ f_W(x, \lambda) - \frac{\pi}{\sqrt{\lambda}} f_M(x, \lambda) & \text{if } x < 0. \end{cases}$$

Now it follows from (5.13), (5.14) and (5.15) that

$$b_{f_M}(\cdot, \lambda) = \begin{pmatrix} 0 \\ 2i\sqrt{\lambda} \\ 0 \\ -2i\sqrt{\lambda} \end{pmatrix}, \quad b_{\tilde{f}_W}(\cdot, \lambda) = \begin{pmatrix} 1 \\ -c_\lambda(1) - i\pi \\ -1 \\ c_\lambda(1) \end{pmatrix}.$$

These vectors span the kernel of the 2×4 matrix $B_{-i\pi}$. Therefore, the operator which was considered in [4] is (up to its sign) $T_{-i\pi}$.

In order to study the asymptotic behaviour of the functions f_M and f_W for $\lambda \rightarrow \infty$, we use the following relations [2, 6.13(1) and (2)] [16, 4.7(2)–(4)]:

$$\begin{aligned} \Phi(\alpha, \beta, z) &= \frac{\Gamma(\beta)e^{\alpha i\pi \operatorname{sgn} \operatorname{Im} z}}{\Gamma(\beta - \alpha)} z^{-\alpha} + \frac{\Gamma(\beta)}{\Gamma(\alpha)} e^z z^{\alpha - \beta} \\ &\quad + O(z^{-\alpha-1}) + O(e^z z^{\alpha - \beta - 1}), \end{aligned} \quad (5.16)$$

$$\Psi(\alpha, \beta, z) = z^{-\alpha} + O(z^{-\alpha-1}) \quad (5.17)$$

if $z \rightarrow \infty$. The expansion (5.16) holds in the sector $-\pi < \arg z < \pi$, the expansion (5.17) in the sector $-3\pi/2 < \arg z < 3\pi/2$. If $x \in \mathbb{R} \setminus \{0\}$ is fixed, then for $\kappa \rightarrow 0$,

$$\Gamma(1 \pm \kappa) = 1 + O(\kappa), \quad e^{\pm i\pi(1-\kappa)} = -1 + O(\kappa), \quad z^\kappa = e^{\kappa(\ln x - \ln \kappa)} = 1 + O(\kappa \ln \kappa),$$

and we find the following asymptotics for $\lambda \rightarrow \infty$ in the sector $-\pi < \arg \lambda < \pi$:

$$f_M(x, \lambda) = e^{-i\sqrt{\lambda}x} - e^{i\sqrt{\lambda}x} + O\left(\frac{\ln \lambda}{\sqrt{\lambda}} e^{|x \operatorname{Im} \sqrt{\lambda}|}\right), \quad (5.18)$$

$$f_W(x, \lambda) = e^{i\sqrt{\lambda}x} + O\left(\frac{\ln \lambda}{\sqrt{\lambda}} e^{i\sqrt{\lambda}x}\right). \quad (5.19)$$

THEOREM 5.3. *If $\gamma \in \mathbb{C}$, then the spectrum $\sigma(T_\gamma)$ consists of isolated normal eigenvalues λ_n , $n \in \mathbb{N}$, of geometric multiplicity one, and all but finitely many of them are simple. If they are numbered according to non-decreasing absolute value, then the following asymptotic formula holds:*

$$\lambda_n = \frac{\pi^2 n^2}{(b-a)^2} + O(\ln n) \quad \text{for } n \rightarrow \infty. \quad (5.20)$$

In the proof of the theorem we use the following lemma.

LEMMA 5.4. *An entire function $F(z)$ of the form*

$$F(z) = \sin z + O\left(\frac{\ln z}{z} \exp |\operatorname{Im} z|\right) \quad \text{for } |z| \rightarrow \infty$$

has infinitely many zeros and all but finitely many of them are simple. For $n \in \mathbb{Z}$ with $|n|$ sufficiently large, there is a disc of radius

$$\rho_n = O\left(\frac{\ln |n|}{n}\right) \quad \text{for } |n| \rightarrow \infty,$$

around the point $n\pi$ which contains exactly one zero of $F(z)$; outside these discs lie only finitely many zeros of $F(z)$.

Proof. Since $F(z)$ is entire and does not vanish identically, its zeros are countable and have no accumulation point in \mathbb{C} . We consider the zeros only in the right half-plane; the zeros in the left half-plane can be treated similarly. There exist positive real numbers r, C_1, C_2 such that for the zeros $\zeta = s + it$, with $|\zeta| > r$,

$$|\sinh t| \leq |\sin \zeta| \leq C_1 \left| \frac{\ln \zeta}{\zeta} \right| \exp |t| \leq 2C_1 \left| \frac{\ln \zeta}{\zeta} \right| (|\sinh t| + 1).$$

Hence

$$1 + \frac{1}{|\sinh t|} \geq \frac{1}{C_2} \left| \frac{\zeta}{\ln \zeta} \right|, \quad |\zeta| > r,$$

which implies that all zeros ζ lie in a strip $|t| \leq C$ with $C > 0$, and with $C_3 = C_1 \exp C$,

$$|\sin \zeta| \leq C_3 \left| \frac{\ln \zeta}{\zeta} \right|, \quad |\zeta| > r.$$

Denote by R_n , $n \in \mathbb{N}$, the rectangle

$$n\pi - \pi/2 \leq \operatorname{Re} z \leq n\pi + \pi/2, \quad -C \leq \operatorname{Im} z \leq C.$$

Then

$$m := \min_{z \in \partial R_n} |\sin z| > 0$$

and for sufficiently large n , say $n \geq n_0$,

$$|F(z) - \sin z| < m \leq |\sin z|, \quad z \in \partial R_n.$$

Rouché's theorem implies that $F(z)$, like $\sin z$, has exactly one zero in R_n for $n \geq n_0$ and that this zero is simple. We now claim that for n sufficiently large, the zero of $F(z)$ in R_n lies in a circle of radius $\rho_n = O(n^{-1} \ln n)$ around the zero $z = n\pi$ of $\sin z$. To prove the claim, first choose $\rho > 0$ such that the inequality

$$|\sin z| \geq \frac{1}{2}|z - \pi n|$$

holds for all $n \in \mathbb{N}$ and $|z - \pi n| \leq \rho$. Then choose C_4 such that

$$|F(z) - \sin z| < \frac{C_4}{2} \left(\frac{\ln n}{n} \right), \quad z \in R_n, \quad n \geq 2.$$

Finally, choose $n_1 \geq \max(2, n_0)$ so large that $\rho_n := C_4(n^{-1} \ln n) < \rho$ for all $n \geq n_1$. Then, for $n \geq n_1$ and $|z - \pi n| = \rho_n$,

$$|F(z) - \sin z| < \frac{1}{2}\rho_n = \frac{1}{2}|z - \pi n| \leq |\sin z|.$$

The claim now follows again from Rouché's theorem. \square

Proof of theorem 5.3. We consider the matrix B_γ from (4.2) and the corresponding 2×2 matrix function M_γ defined by (5.8). Since $M_\gamma(\lambda)$, $\lambda \in \mathbb{C}$, is never the zero matrix, the geometric multiplicity of the eigenvalues of T_γ is one; see theorem 5.1. A straightforward calculation shows that, up to a non-zero factor, the determinant $\det M_\gamma(\lambda)$ equals

$$\begin{aligned} & \begin{vmatrix} f_M(a, \lambda) & f_M(b, \lambda) \\ -2i\sqrt{\lambda}f_W(a, \lambda) + (i\pi - c_\lambda(1))f_M(a, \lambda) & -2i\sqrt{\lambda}f_W(b, \lambda) + (\gamma - c_\lambda(1))f_M(b, \lambda) \end{vmatrix} \\ &= \begin{vmatrix} f_M(a, \lambda) & f_M(b, \lambda) \\ -2i\sqrt{\lambda}f_W(a, \lambda) + i\pi f_M(a, \lambda) & -2i\sqrt{\lambda}f_W(b, \lambda) + \gamma f_M(b, \lambda) \end{vmatrix}. \end{aligned}$$

Now, relations (5.18) and (5.19) imply that this determinant for $\lambda \rightarrow \infty$ asymptotically behaves like

$$\begin{aligned} & -2i\sqrt{\lambda} \begin{vmatrix} f_M(a, \lambda) & f_M(b, \lambda) \\ f_W(a, \lambda) & f_W(b, \lambda) \end{vmatrix} + O(e^{|(b-a) \operatorname{Im} \sqrt{\lambda}|}) \\ &= -2i\sqrt{\lambda} \begin{vmatrix} e^{-i\sqrt{\lambda}a} - e^{i\sqrt{\lambda}a} & e^{-i\sqrt{\lambda}b} - e^{i\sqrt{\lambda}b} \\ e^{-i\sqrt{\lambda}a} & e^{-i\sqrt{\lambda}b} \end{vmatrix} + O(e^{|(b-a) \operatorname{Im} \sqrt{\lambda}|} \ln \lambda) \\ &= 4\sqrt{\lambda}((b-a)\sqrt{\lambda}) + O(e^{|(b-a) \operatorname{Im} \sqrt{\lambda}|} \ln \lambda). \end{aligned}$$

If we put $\zeta = (b-a)\sqrt{\lambda}$, apply lemma 5.4, and observe again theorem 5.1, then the claim follows. \square

6. Basis properties of the root vectors of T_γ

Recall that a sequence (f_n) , $n \in \mathbb{N}$, of elements of a separable Hilbert space \mathcal{H} is called a *basis* of \mathcal{H} if each $y \in \mathcal{H}$ has a unique representation

$$y = \sum_{n=1}^{\infty} c_n f_n, \quad \text{with } c_n \in \mathbb{C}, \quad n \in \mathbb{N},$$

where the sum converges in the norm of \mathcal{H} . The basis (f_n) , $n \in \mathbb{N}$, of \mathcal{H} is called a *Bari basis* if it is quadratically close to an orthonormal basis $\{e_n: n \in \mathbb{N}\}$ of \mathcal{H} , which means that

$$\sum_{n=1}^{\infty} \|f_n - e_n\|^2 < \infty.$$

For this notion and its properties, see, for example, [9, ch. VI]. We use the following criterion about the existence of a Bari basis [9, theorem VI.4.1]:

CRITERION. *Let T be a bounded dissipative operator in a Hilbert space such that $T - T^*$ is compact. Denote by μ_n , $n \in \mathbb{N}$, the mutually different eigenvalues of T and by l_n the geometric multiplicity of μ_n , and suppose that*

$$\sum \min(l_n, l_m) \frac{\operatorname{Im} \mu_n \operatorname{Im} \mu_m}{|\mu_n - \mu_m|^2} < \infty, \quad (6.1)$$

where the sum runs over all $n, m \in \mathbb{N}$ such that $n \neq m$ and $\operatorname{Im} \mu_n \neq 0$, $\operatorname{Im} \mu_m \neq 0$. If we choose in each eigenspace of T an orthonormal basis, then the sequence of all these basis elements forms a Bari basis in its closed linear hull.

We also use the well-known result of Lidskii [9, theorem V.2.3]:

RESULT. *A dissipative trace class operator has a complete system of root vectors.*

If γ is real or ∞ , then the operator T_γ is self-adjoint. By an argument as in the proof of the following theorem, it follows that its resolvent is a trace class operator. Hence T_γ , $\gamma \in \mathbb{R} \cup \{\infty\}$, has an orthonormal basis of eigenfunctions. The main result of this section is the following theorem.

THEOREM 6.1. *If $\gamma \in \mathbb{C}^+ \cup \mathbb{C}^-$, then the root vectors of T_γ can be chosen to form a Bari basis of $\mathcal{L}^2(I)$.*

Proof. Let $l \in \rho(T_\gamma) \cap \rho(T_0)$ be a real number. The spectral mapping theorem and theorem 5.3 imply that the eigenvalues η_n , $n \in \mathbb{N}$, of $(T_\gamma - l)^{-1}$ satisfy the relation

$$\eta_n = \frac{1}{cn^2 + O(\ln n)} = \frac{1}{cn^2} + O\left(\frac{\ln n}{n^4}\right) \quad \text{for } n \rightarrow \infty \quad (6.2)$$

with $c := \pi^2(b-a)^{-2}$. By theorem 4.3, T_0 is self-adjoint, hence also $(T_0 - l)^{-1}$ is self-adjoint, and since its eigenvalues satisfy relation (6.2), it is a trace class operator. If $\gamma \neq 0$, the difference $(T_\gamma - l)^{-1} - (T_0 - l)^{-1}$ is one-dimensional and therefore also $(T_\gamma - l)^{-1}$ is a trace class operator.

In order to prove that the root vectors of T_γ form a Bari basis, we suppose that $\gamma \in \mathbb{C}^+$; the case $\gamma \in \mathbb{C}^-$ can be treated analogously. The operator $-(T_\gamma - l)^{-1}$ is

dissipative and a trace class operator. Therefore, the closed linear span of its root vectors is the whole space $\mathcal{L}^2(I)$. Next we verify that the eigenvalues of $(T_\gamma - I)^{-1}$, which we denote by η_n , satisfy condition (6.1). Since the algebraic multiplicity of all but finitely many eigenvalues is one by theorem 5.3, this condition simplifies to

$$\sum_{1 \leq m < n} \frac{\operatorname{Im} \eta_m \operatorname{Im} \eta_n}{|\eta_m - \overline{\eta_n}|^2} < \infty. \quad (6.3)$$

Relation (6.2) implies for $1 \leq m < n$ and suitable constants C_1, C_2, C_3 that

$$\frac{\operatorname{Im} \eta_m \operatorname{Im} \eta_n}{|\eta_m - \overline{\eta_n}|^2} \leq C_1 \frac{\ln m \ln n}{|(n-m)(n+m) - C_1(\ln m + \ln n)|^2} \leq C_3 \frac{(\ln(m+n))^2}{(n-m)^2(n+m)^2},$$

here we have used the inequalities $\ln n, \ln m \leq \ln(n+m)$ and the fact that

$$(n-m)^{-1}(n+m)^{-1}(\ln m + \ln n) \rightarrow 0 \quad \text{if } m < n, \quad n \rightarrow \infty.$$

Since for sufficiently large x the function $x^{-1} \ln x$ is decreasing, then with $k = n-m$ and some constant C_4 , we finally obtain

$$\sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{(\ln(2m+k))^2}{k^2(2m+k)^2} \leq C_4 \sum_{n=1}^{\infty} \frac{(\ln 2m)^2}{(2n)^2} \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

□

If $\gamma \in \mathbb{C}^+ \cup \mathbb{C}^-$, then T_γ or $-T_\gamma$ is dissipative and it is easy to see that the relation

$$T_\gamma^* = T_{\overline{\gamma}}$$

holds. Denote by $(\lambda)_n$, $n \in \mathbb{N}$, the sequence of (mutually different) eigenvalues of T_γ , and denote by

$$g_{n,1}, g_{n,2}, \dots, g_{n,m_n}$$

a basis of the root subspace of T_γ corresponding to λ_n , such that the system of all elements $g_{n,k}$, $k = 1, 2, \dots, m_n$, $n \in \mathbb{N}$, is a Bari basis of $\mathcal{L}^2(I)$. Then the complex conjugate functions

$$\overline{g_{n,1}}, \overline{g_{n,2}}, \dots, \overline{g_{n,m_n}}$$

form a basis of the root subspace of $T_{\overline{\gamma}} = T_\gamma^*$ corresponding to $\overline{\lambda_n}$. We introduce for $n \in \mathbb{N}$ the $m_n \times m_n$ matrix

$$G_n := \begin{pmatrix} (g_{n,1}, \overline{g_{n,1}}) & \dots & (g_{n,m_n}, \overline{g_{n,1}}) \\ \vdots & & \vdots \\ (g_{n,1}, \overline{g_{n,m_n}}) & \dots & (g_{n,m_n}, \overline{g_{n,m_n}}) \end{pmatrix}.$$

The root subspaces of T_γ at λ_n and of $T_{\overline{\gamma}}^*$ at $\overline{\lambda_m}$ are orthogonal if $m \neq n$, and are in duality if $m = n$. Hence the matrix G_n is invertible. For $y \in \mathcal{L}^2(I)$ we define numbers $c_{n,k}$, $k = 1, 2, \dots, m_n$, $n \in \mathbb{N}$, by the relation

$$\begin{pmatrix} c_{n,1}(y) \\ \vdots \\ c_{n,m_n}(y) \end{pmatrix} := G_n^{-1} \begin{pmatrix} (y, \overline{g_{n,1}}) \\ \vdots \\ (y, \overline{g_{n,m_n}}) \end{pmatrix}. \quad (6.4)$$

THEOREM 6.2. If $\gamma \in \mathbb{C} \setminus \mathbb{R}$, then each element $y \in \mathcal{L}^2(I)$ admits the following unique expansion,

$$y = \sum_{n=1}^{\infty} \sum_{k=1}^{l_n} c_{n,k}(y) g_{n,k}, \quad (6.5)$$

where the left sum converges in the norm of $\mathcal{L}^2(I)$.

Proof. For $y = g_{n_0,l}$ with $1 \leq l \leq m_{n_0}$, the expansion (6.5) follows from the definitions of the matrix G_n and of the coefficients $c_{n,k}(y)$ and from the fact that $c_{n,k}(g_{n_0,l}) = 0$ if $n \neq n_0$. For arbitrary $y \in \mathcal{L}^2(I)$ it is now a consequence of the properties of a Bari basis. \square

If the elements $g_{n,k}$, $k = 1, 2, \dots, m_k$, which span the root subspace of T_γ at λ_n are chosen to form a Jordan chain:

$$(T_\gamma - \lambda_n)g_{n,1} = 0, \quad (T_\gamma - \lambda_n)g_{n,2} = g_{n,1}, \quad (T_\gamma - \lambda_n)g_{n,m_n} = g_{n,m_n-1},$$

then the elements $\overline{g_{n,k}}$, $k = 1, 2, \dots, m_n$, form a Jordan chain of T_γ^* at $\overline{\lambda_n}$ and we get

$$\begin{aligned} (g_{n,k}, \overline{g_{n,l}}) &= ((T_\gamma - \lambda_n)^{m_n-k} g_{n,m_n}, (T_\gamma^* - \overline{\lambda_n})^{m_n-l} \overline{g_{n,m_n}}) \\ &= ((T_\gamma - \lambda_n)^{2m_n-(k+l)} g_{n,m_n}, \overline{g_{n,m_n}}). \end{aligned}$$

Therefore, the matrix G_n is now a Hankel matrix and right lower triangular. Since G_n is invertible, the numbers $(g_{n,k}, \overline{g_{n,l}})$ with $k + l = m_n$ are not zero. Now it is easy to see that the Jordan chain $g_{n,k}$, $k = 1, 2, \dots, m_n$, can be modified such that the matrix G_n becomes $(g_{n,1}, \overline{g_{n,m_n}})$ times the m_n -sip matrix $(\delta_{k,m_n-l+1})_{k,l=1}^{m_n}$ [10, theorem I.3.3]. Indeed, replace the Jordan chain $g_{n,k}$ by a Jordan chain $g'_{n,k}$, the last element of which has the form $g'_{n,k} = \sum_{l=1}^{m_n} \alpha_k g_{n,l}$, and determine the α_k such that $(g'_{n,k}, \overline{g'_{n,l}}) = \delta_{k,m_n-l+1}$, $k, l = 1, 2, \dots, m_n$. With this choice of the Jordan chains at all the eigenvalues λ_n of T_γ , expansion (6.5) simplifies to

$$y = \sum_{n=1}^{\infty} \sum_{k=1}^{m_n} (y, \overline{g_{n,m_n-k+1}}) g_{n,k}.$$

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